$\mathbb{A}^1$-homotopy theory and vector bundles

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Outline

1. Vector bundles
   - Topological vector bundles
   - Algebraic vector bundles

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Real vector spaces

Let $\mathbb{R}$ be the field of real numbers. Every $\mathbb{R}$-module (a.k.a. $\mathbb{R}$-vector space) admits not only a generating set, but even a basis. (Linear Algebra, first year)

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In particular, the dimension function induces a bijection

$$\{\text{finitely generated } \mathbb{R} - \text{modules}\}/\text{isomorphism} \xrightarrow{\cong} \mathbb{N}$$

which is compatible with (direct) sum and (tensor) product.

Topological vector bundles

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- a continuous map $p\colon V \longrightarrow X$
- an open cover $\{U_\alpha \hookrightarrow X\}_\alpha$ and isomorphisms $f_\alpha\colon p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^n$ over U_α such that
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Same works with $\mathbb{C}$ in place of $\mathbb{R}$.

Topological vector bundles

Examples:

- A real vector bundle is *trivial* if it is isomorphic to the projection $\mathrm{pr}\colon X \times \mathbb{R}^n \longrightarrow X$.
- The tangent bundle of a smooth manifold of dimension d is a real vector bundle of rank d.
- The map $\mathbb{RP}^{d+1} \setminus \{(0\colon \cdots : 0\colon 1)\} \longrightarrow \mathbb{RP}^d$ forgetting the last homogeneous coordinate is a nontrivial line bundle.

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Theorem (Bott-Milnor, Kervaire 1958)

The tangent bundle of the sphere S^d is trivial if and only if $d \in \{0, 1, 3, 7\}$.

Grassmannians

Let $\mathrm{Gr}_{\mathbb{R}}(n, k)$ be the space of sub-$\mathbb{R}$-vector spaces of $\mathbb{R}^{n+k}$ having dimension n. It can also be described as the homogeneous space

$$\mathrm{Gr}_{\mathbb{R}}(n, k) \cong O(n+k)/O(n) \times O(k)$$

for the orthogonal group. For example, $\mathbb{RP}^d = \mathrm{Gr}_{\mathbb{R}}(1, d) \cong \mathrm{Gr}_{\mathbb{R}}(d, 1)$.

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$\mathbb{RP}^d = \mathrm{Gr}_{\mathbb{R}}(1,d) \cong \mathrm{Gr}_{\mathbb{R}}(d,1)$.
It comes with a tautological vector bundle of rank n

$$\begin{array}{ccc} \gamma_{n,k} & \hookrightarrow & \mathrm{Gr}_{\mathbb{R}}(n,k) \times \mathbb{R}^{n+k} \\ & \searrow & \downarrow \mathrm{pr} \\ & & \mathrm{Gr}_{\mathbb{R}}(n,k) \end{array}$$

(a subbundle of the trivial vector bundle of rank $n+k$) whose fiber at a point $V \hookrightarrow \mathbb{R}^{n+k}$ is the vector space V.

Classifying topological vector bundles

Let $\mathrm{Gr}_{\mathbb{R}}(n, \infty)$ be the colimit of the Grassmannians $\mathrm{Gr}_{\mathbb{R}}(n, k)$ induced by the inclusion $\mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}, x \longmapsto (x, 0)$. The tautological rank n vector bundle extends accordingly: $\gamma_{n,\infty} \longrightarrow \mathrm{Gr}_{\mathbb{R}}(n, \infty)$.

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Theorem (Steenrod 1951)

Let X be a paracompact Hausdorff space. Sending a map $f\colon X \longrightarrow \mathrm{Gr}_{\mathbb{R}}(n, \infty)$ to the pullback vector bundle $f^*\gamma_{n,\infty}$ defines a bijection

$$[X, \mathrm{Gr}_{\mathbb{R}}(n, \infty)] \cong \{\text{rank } n \text{ vector bundles over } X\}/\text{isomorphism}$$

from the set of homotopy classes of maps to the set of isomorphism classes of vector bundles of rank n.

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Because of Steenrod's theorem, the space $\mathrm{Gr}_{\mathbb{R}}(n,\infty)$ is a *classifying space* for rank n vector bundles.

Classifying topological vector bundles

Since $\mathrm{Gr}_{\mathbb{R}}(1, \infty) = \mathbb{RP}^\infty$ satisfies

$$\pi_n \mathbb{RP}^\infty = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1 \\ \{1\} & n \neq 1 \end{cases}$$

real line bundles are easily classified. For example, S^1 admits two distinct line bundles up to isomorphism, and S^2 just one.

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real line bundles are easily classified. For example, S^1 admits two distinct line bundles up to isomorphism, and S^2 just one. One may also compute

$$\pi_2 \mathrm{Gr}_{\mathbb{R}}(2, \infty) = \pi_1 O(2) \cong \mathbb{Z}$$

whence real vector bundles of rank 2 on S^2 correspond bijectively to integers.

The Serre-Swan correspondence

To every topological space X one can associate the ring $\mathbb{R}(X)$ of real-valued maps on X.

Theorem (Swan 1962)

Sending a real vector bundle $p\colon V \longrightarrow X$ to the $\mathbb{R}(X)$-module of sections $X \longrightarrow V$ of p defines a bijection

$$\{\text{vector bundles over } X\}/\text{iso} \cong \{\text{f. g. projective } \mathbb{R}(X)\text{−modules}\}/\text{iso}$$

if X is compact Hausdorff.

In particular, trivial vector bundles over X then correspond precisely to finitely generated free $\mathbb{R}(X)$-modules.

Outline

Projective modules

Let R be any (commutative unital) ring. Every R-module admits a generating set, but not necessarily a basis.

- $R = \mathrm{Mat}_{2\times 2}(\mathbb{R})$, $M = \mathbb{R}^2$
- $R = \mathbb{Z}[\sqrt{-5}]$, $M = (2, 1-\sqrt{-5})$
- $R = \mathbb{R}(\mathbb{RP}^1)$,
  $M = \{f\colon \mathbb{RP}^1 \longrightarrow \mathbb{RP}^2 \setminus \{(0:0:1)\} | p \circ f = \mathrm{id}\}$ where p forgets the last homogeneous coordinate

Direct summands of free R-modules are called *projective*. The smallest example of a non-projective module is the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$.

Projective modules

Question

Can one describe the set of finitely generated projective R-modules up to isomorphism?

Answer

Sometimes. It is $\mathbb{N}$ in each of the following cases:

- *R is a principal ideal domain*
- *R is local (Kaplansky 1958)*
- *R is a polynomial ring over a principal ideal domain (Quillen, Suslin 1976)*

Projective modules as vector bundles

Finitely generated projective modules over a ring R can be viewed as vector bundles over a space. The space in question is the *affine variety* $\mathrm{Spec}(R)$, whose points are prime ideals of R, and whose topology is the Zariski topology:

Projective modules as vector bundles

Finitely generated projective modules over a ring R can be viewed as vector bundles over a space. The space in question is the *affine variety* $\mathrm{Spec}(R)$, whose points are prime ideals of R, and whose topology is the Zariski topology: A subset $C \subset \mathrm{Spec}(R)$ is *closed* if and only if there is an ideal $I \subset R$ with

$$C = \{P \in \mathrm{Spec}(R) | I \subset P\}.$$

For example, the closed subsets in $\mathrm{Spec}(\mathbb{Z})$ are precisely the finite subsets and $\mathrm{Spec}(\mathbb{Z})$. Closed points in $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\big(\mathbb{C}[t]\big)$ correspond to linear polynomials $t - z$ with $z \in \mathbb{C}$, hence to complex numbers.

Non-affine varieties

In order to construct algebraic analogues of Grassmannians, generalize the notion of "affine variety" to "variety" (a.k.a. "scheme"): A *variety* is a locally ringed topological space which admits an open cover by affine varieties.

Example

The *projective line* over R is obtained by gluing two affine lines $\mathbb{A}^1_R = \operatorname{Spec}\big(R[x]\big)$ as follows:

$$\begin{array}{ccc}
\mathbb{A}^1_R \setminus \{0\} = \operatorname{Spec}\big(R[t, t^{-1}]\big) & \hookrightarrow & \operatorname{Spec}\big(R[t]\big) = \mathbb{A}^1_R \\
\downarrow & & \downarrow \\
\operatorname{Spec}\big(R[t^{-1}]\big) \cong \mathbb{A}^1_R & \hookrightarrow & \mathbb{P}^1
\end{array}$$

Projective spaces

Example

More generally, the *projective space* $\mathbb{P}^d_R$ over R of dimension d is obtained by suitably gluing $d+1$ affine spaces $\mathbb{A}^d_R = \mathrm{Spec}\big(R[t_1, \ldots, t_d]\big)$ of dimension d.

A different description arises as the homogeneous space:

$$\mathbb{P}^d = \mathrm{GL}_{d+1}/H$$

where H is a parabolic subgroup with Levi factor $\mathrm{GL}_d \times \mathrm{GL}_1$. Here $\mathrm{GL}_n = \mathrm{Spec}\big(\mathbb{Z}[t_{11}, t_{12}, \ldots, t_{1n}, t_{21}, \ldots, t_{nn}, \det^{-1}]\big)$ is the variety of $n \times n$ matrices whose determinant is non-zero, with group structure given by multiplication of matrices.

Trivial vector bundles

Trivial vector bundles over $X = \mathrm{Spec}(R)$ are of the form

$$\mathbb{A}^n_R = \mathrm{Spec}(R[t_1, \ldots, t_n]) \xrightarrow{\mathrm{Spec}(\text{inclusion})} \mathrm{Spec}(R) = X$$

where $R[t_1, \ldots, t_n]$ is the ring of polynomials in n variables with coefficients in R.
The set of sections $\mathrm{Spec}(R) \longrightarrow \mathbb{A}^n_R$ of the projection $\mathbb{A}^n_R \longrightarrow \mathrm{Spec}(R)$ constitutes a free R-module on n generators.

Algebraic vector bundles

Let X be a variety. A *vector bundle* of rank n over X is a

- map $p\colon V \longrightarrow X$ of varieties,
- an open cover $\{U_\alpha \hookrightarrow X\}_\alpha$ and isomorphisms $f_\alpha\colon p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{A}^n$ over U_α such that
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Sending a vector bundle $p\colon V \longrightarrow X = \mathrm{Spec}(R)$ to the R-module of sections $X \longrightarrow V$ defines a bijection between vector bundles over $X = \mathrm{Spec}(R)$ and finitely generated projective R-modules up to isomorphism.

Vector bundles over the projective line

Theorem (Grothendieck 1957)

Let F be an algebraically closed field. Every vector bundle over $\mathbb{P}^1_F$ is a direct sum on line bundles in a unique way. If $\mathcal{O}_{\mathbb{P}^1_F}(-1)$ denotes the tautological line bundle, the map

$$n \longmapsto \mathcal{O}_{\mathbb{P}^1_F}(-n) := \left(\mathcal{O}_{\mathbb{P}^1_F}(-1)\right)^{\otimes n}$$

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Theorem (Barth 1977)

The moduli space of stable rank 2 vector bundles $V \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ with $\wedge^2 V$ trivial and fixed second Chern class $c_2(V) = n > 1$ is smooth, rational and connected of dimension $4n - 3$.

Smooth varieties

Let F be a field. Recall that an affine F-variety $X = \mathrm{Spec}(R)$ is smooth if R is an F-algebra of the form

$$F[t_1, \ldots, t_n]/(p_1, \ldots, p_r)$$

where $p_1, \ldots, p_r$ are polynomials such that the Jacobian

$$\left(\frac{\partial p_k}{\partial t_\ell}(\rho)\right)_{k,\ell}$$

has rank $n - \dim R_\rho$ for every closed point $\rho \in \mathrm{Spec}(R)$.

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has rank $n - \dim R_\rho$ for every closed point $\rho \in \mathrm{Spec}(R)$.
A smooth (affine) F-variety of dimension d has a tangent vector bundle of rank d, obtained via Kähler differentials.

A classifying “space” for algebraic vector bundles

Let $\mathrm{Gr}(n,k)$ be the Grassmann variety of linear subspaces of dimension n in a linear space of dimension $n+k$. It comes with a tautological vector bundle of rank n

$$\begin{array}{ccc} \gamma_{n,k} & \hookrightarrow & \mathrm{Gr}(n,k)\times\mathbb{A}^{n+k} \\ & \searrow & \downarrow \mathrm{pr} \\ & & \mathrm{Gr}(n,k) \end{array}$$

(a subbundle of the trivial vector bundle of rank $n+k$) whose fiber at a point $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+k}$ is $\mathbb{A}^n$.
Let $\mathrm{Gr}(n,\infty)$ be the colimit of the Grassmannians $\mathrm{Gr}(n,k)$ induced by the inclusion $\mathbb{A}^{n+k} \hookrightarrow \mathbb{A}^{n+k+1}, x \longmapsto (x,0)$.

A classifying “space” for algebraic vector bundles

Theorem (Morel 2012)

Let F be a field, and let $X = \mathrm{Spec}(R)$ be a smooth affine F-variety. Sending a map $f\colon X \longrightarrow \mathrm{Gr}(n,\infty)$ to the pullback vector bundle $f^*\gamma_{n,\infty}$ defines a bijection

$$[X, \mathrm{Gr}(n,\infty)]_{\mathbb{A}^1} \cong \{\text{rank } n \text{ vector bundles over } X\}/\text{isomorphism}$$

from the set of $\mathbb{A}^1$-homotopy classes of maps to the set of isomorphism classes of vector bundles of rank n.

Outline

What are $\mathbb{A}^1$-homotopy classes of maps?

Central idea: Do homotopy theory for varieties, with the affine line $\mathbb{A}$ as “interval” parametrizing homotopies.

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Why? Because homotopy theory played an important role in the study and classification of (smooth) manifolds.

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Central idea: Do homotopy theory for varieties, with the affine line $\mathbb{A}$ as "interval" parametrizing homotopies.
Why? Because homotopy theory played an important role in the study and classification of (smooth) manifolds.
More specifically, assuming the homotopy theory for varieties is set up properly, the following theorems hold in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category $\mathbf{Ho}(F)$ of a field F.

Reasons for $\mathbb{A}^1$-homotopy theory

Theorem (Morel-Voevodsky 1999, Homotopy Purity)

Let $i\colon Z \hookrightarrow X$ be a closed embedding of smooth F-varieties, with normal vector bundle $Ni \longrightarrow Z$. Let $z\colon Z \longrightarrow Ni$ be its zero section. In $\mathbf{Ho}(F)$ there is a canonical isomorphism:

$$X/X \setminus i(Z) \cong Ni/Ni \setminus z(Z)$$

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$$X/X \smallsetminus i(Z) \cong Ni/Ni \smallsetminus z(Z)$$

The Homotopy Purity Theorem supplies an analog of tubular neighborhoods, as in differential topology: If $i\colon Z \hookrightarrow X$ is a smooth submanifold, the normal bundle of i embeds in X such that i corresponds to the zero section. In particular, the Thom space of the normal bundle of i is homotopy equivalent to the quotient $X/X \smallsetminus Ni$.

Reasons for $\mathbb{A}^1$-homotopy theory

Theorem (Morel-Voevodsky 1999)

Let $\mathrm{Gr}(\infty, \infty)$ be the infinite Grassmann variety, considered as an object in $\mathbf{Ho}(F)$. For every smooth F-variety X, there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ho}(F)}\big(X, \mathrm{Gr}(\infty, \infty) \times \mathbb{Z}\big) \cong K^0(X)$$

to the Grothendieck group of vector bundles on X.

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In fact, $\mathrm{Gr}(\infty, \infty) \times \mathbb{Z}$ also represents Quillen's higher algebraic K-groups, by mapping from $S^n \wedge X_+$ in the pointed homotopy category $\mathbf{Ho}_\bullet(F)$.

Reasons for $\mathbb{A}^1$-homotopy theory

Theorem (Voevodsky 1998)

Let $n \in \mathbb{N}$. There exists an object $K(\mathbb{Z}, n) \in \mathbf{Ho}(F)$ and a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ho}(F)}\big(X, K(\mathbb{Z}, n)\big) \cong CH^n(X)$$

to the Chow group of cycles of codimension n.

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to the Chow group of cycles of codimension n.

In fact, $K(\mathbb{Z}, n)$ also represents Bloch's higher Chow groups, a.k.a. motivic cohomology, of weight n.

Outline

Setting up $\mathbb{A}^1$-homotopy theory

Homotopy theory requires universal constructions:

- classifying objects
- quotients
- suspensions

These are not necessarily available within the category of varieties.

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- classifying objects
- quotients
- suspensions

These are not necessarily available within the category of varieties.
Solution: Yoneda embedding from varieties to presheaves. For technical reasons, presheaves with values in *simplicial* sets have an advantage, since homotopy theory of simplicial sets is already available.

Simplicial presheaves

Let F be a field. A *space over F* is a functor

$$\mathrm{Sm}_F^{\mathrm{op}} \longrightarrow \mathbf{sSet}$$

from the category of smooth F-varieties to the category of simplicial sets.

Simplicial presheaves

Let F be a field. A *space over* F is a functor

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from the category of smooth F-varieties to the category of simplicial sets.

Example

If L is a simplicial set, the constant functor $\mathrm{Sm}_F^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ with value L is a space over F, also denoted L.
If $X \in \mathrm{Sm}_F$ is a smooth F-variety, the representable functor

$$\begin{aligned} \mathrm{Sm}_F^{\mathrm{op}} &\longrightarrow \mathbf{sSet} \\ Y &\longmapsto \mathrm{Hom}_{\mathrm{Sm}_F}(Y, X) \end{aligned}$$

is a (discrete) space over F, also denoted X.

Maps of spaces over F are natural transformations. Let $\mathbf{Spc}_F$ be the category of spaces over F.

Algebraic K-theory as a space over F

Quillen's definition of algebraic K-theory provides an interesting space over F. Associate to $X \in \mathrm{Sm}_F$ the category $\mathbf{Vect}_X$ of vector bundles over X.

Algebraic K-theory as a space over F

Quillen's definition of algebraic K-theory provides an interesting space over F. Associate to $X \in \mathrm{Sm}_F$ the category $\mathbf{Vect}_X$ of vector bundles over X. It is an exact category in a natural way. Let $\mathrm{Q}\mathbf{Vect}_X$ be Quillen's construction providing a categorical group completion for exact categories. Let $\mathrm{BQ}\mathbf{Vect}_X$ be its classifying space, a simplicial set. Its homotopy groups are the algebraic K-groups of X. In particular, its fundamental group is the Grothendieck group $K^0(X)$ of vector bundles on X.

Algebraic K-theory as a space over F

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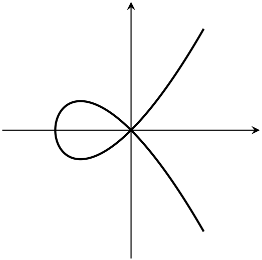
$$K^Q\colon \mathrm{Sm}_F^{\mathrm{op}} \longrightarrow \mathbf{sSet}$$

$$X \longmapsto K^Q = \Omega \mathrm{Ex}^\infty \mathrm{BQ}\mathbf{Vect}_X$$

is a space over F via pullback of vector bundles. Here Ω denotes the simplicial set of pointed loops, and Ex^∞ is Dan Kan's fibrant replacement functor.

Why only values on smooth F-varieties?

One reason for restricting to smooth F-varieties comes from the desire that varieties should locally look like $\mathbb{A}^d$. This is not the case in general.



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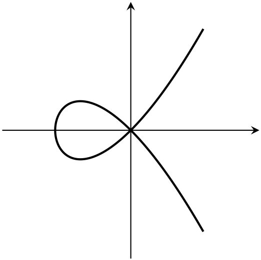

Theorem

Let $f\colon X \longrightarrow \operatorname{Spec}(F)$ be smooth at $x \in X$. Then there exists a Zariski open subset $x \in U \hookrightarrow X$ and a morphism $g\colon U \longrightarrow \mathbb{A}^d_F$ étale at x, such that the equality

$$f|_U = U \xrightarrow{g} \mathbb{A}^d_F \xrightarrow{\mathrm{pr}} \operatorname{Spec}(F)$$

holds.

Objectwise homotopy theory

Let $f\colon A \longrightarrow B$ be a map of spaces over F.

- It is an *objectwise equivalence* if $f(X)$ is a weak equivalence of simplicial sets for every $X \in \mathrm{Sm}_F$.
- It is an *objectwise fibration* if $f(X)$ is a Kan fibration of simplicial sets for every $X \in \mathrm{Sm}_F$.

Outline

Objectwise homotopy theory

Theorem

Objectwise equivalences and objectwise fibrations are part of a model structure on $\mathbf{Spc}_F$*. The model structure is simplicial, combinatorial and monoidal.*

Objectwise homotopy theory

Theorem

Objectwise equivalences and objectwise fibrations are part of a model structure on $\mathbf{Spc}_F$. *The model structure is simplicial, combinatorial and monoidal.*

This is the *projective model structure*. Proof follows by adjointness. The *projective cofibrations* are generated by representable cells:

$$\{X \times (\partial\Delta^n \hookrightarrow \Delta^n)\}_{n\in\mathbb{N}, X\in \mathrm{Sm}_F}$$

So $f\colon A \longrightarrow B$ is a projective cofibration if it is a retract of a sequential composition in which every map is a cobase change of a coproduct of representable cells.

Local homotopy theory

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Local homotopy theory

Objectwise homotopy theory ...

- ... is way too fine. For example, two smooth F-varieties are objectwise equivalent if and only if they are isomorphic.
- ... ignores the geometry of smooth F-varieties completely: The canonical map

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is not an objectwise equivalence.

Local homotopy theory

Objectwise homotopy theory ...

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is not an objectwise equivalence.

Since domain and codomain of f are discrete, it suffices to show that $f(X)$ is not bijective for some $X \in \mathrm{Sm}_F$. However, $\mathrm{id}_{\mathbb{P}^1}$ is not in the image of $f(\mathbb{P}^1)$, since every morphism $\mathbb{P}^1 \longrightarrow \mathbb{A}^1$ is constant.

Outline

Nisnevich topology

Impose a Grothendieck topology homotopy-theoretically (Jardine, Joyal). The choice here is the *Nisnevich topology*.

Nisnevich topology

Impose a Grothendieck topology homotopy-theoretically (Jardine, Joyal). The choice here is the *Nisnevich topology*. A *Nisnevich square* is a pullback square

$$\begin{array}{ccc} V = f^{-1}(U) & \hookrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xrightarrow{j} & X \end{array}$$

of smooth F-varieties in which j is an open embedding and f is an étale morphism inducing an isomorphism

$$(Y \setminus V)_{\mathrm{red}} \xrightarrow{\cong} (X \setminus U)_{\mathrm{red}}$$

on reduced closed complements.

The Nisnevich topology

A Nisnevich square supplies a Nisnevich covering $\{U \hookrightarrow X, Y \longrightarrow X\}$. These generate the Nisnevich topology on Sm_F. So one may talk about (simplicial) Nisnevich sheaves on Sm_F.

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Example

Let $X \in \mathrm{Sm}_F$ and let $j\colon U \hookrightarrow X$ and $f\colon Y \hookrightarrow X$ be both open embeddings. The pullback is the intersection $V = U \cap Y$. The morphism f induces an isomorphism on reduced closed complements $Y \smallsetminus V \longrightarrow X \smallsetminus U$ if and only if $X = U \cup Y$.

In particular, Zariski coverings are Nisnevich coverings.

The Nisnevich topology

Let $f\colon \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ be the morphism induced by the field extension $\mathbb{R} \hookrightarrow \mathbb{C}$. It is étale. Let $x \in \mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}\big(\mathbb{R}[t]\big)$ be the closed point given by the prime ideal $(t^2 + 1)$. It has residue field $\mathbb{C}$.

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$$f^{-1}\big(\mathbb{A}^1_{\mathbb{R}} \smallsetminus \{x\}\big) = \mathbb{A}^1_{\mathbb{C}} \smallsetminus f^{-1}(x) = \mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(t+i),(t-i)\}$$

which shows that

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(t+i),(t-i)\} & \hookrightarrow & \mathbb{A}^1_{\mathbb{C}} \\ \downarrow & & \downarrow f \\ \mathbb{A}^1_{\mathbb{R}} \smallsetminus \{(t^2+1)\} & \overset{j}{\hookrightarrow} & \mathbb{A}^1_{\mathbb{R}} \end{array}$$

is **not** a Nisnevich square.

However, the pullback square

$$\begin{array}{ccc}
\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(t+i),(t-i)\} & \hookrightarrow & \mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(t+i)\} \\
\downarrow & & \downarrow f' \\
\mathbb{A}^1_{\mathbb{R}} \smallsetminus \{(t^2+1)\} & \xrightarrow{j} & \mathbb{A}^1_{\mathbb{R}}
\end{array}$$

is a Nisnevich square. Both reduced closed complements are Spec($\mathbb{C}$), and f' induces the identity.

Why the Nisnevich topology?

The Nisnevich topology (invented by Nisnevich in 1989 as "completely decomposed topology") sits between the Zariski topology and the étale topology. It shares the good properties of both and avoids the bad properties of both.

	Zariski	Nisnevich	étale
smooth implies locally $\mathbb{A}^d$	false	true	true
f_* is exact for f finite	false	true	true
fields are points	true	true	false
cohom. dim. is Krull dim.	true	true	false
K-theory has descent	true	true	false

Nisnevich homotopy theory

Let

$$Q = \begin{array}{ccc} V = f^{-1}(U) & \hookrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xhookrightarrow{j} & X \end{array}$$

be a Nisnevich square in Sm_F. Consider the induced map

$$q \colon Y \coprod_V U \longrightarrow X$$

on simplicial presheaves.

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be a Nisnevich square in Sm_F. Consider the induced map

$$q\colon Y \coprod_V U \longrightarrow X$$

on simplicial presheaves. Factor it using the simplicial mapping cylinder as a projective cofibration

$$q'\colon Y \coprod_V U \rightarrowtail \big((Y \coprod_V U) \times \Delta^1\big) \coprod_{\left((Y \coprod_V U)\times\Delta^0\right)} X =: X'$$

followed by a simplicial homotopy equivalence.

Nisnevich homotopy theory

Consider the diagram (with canonical horizontal maps)

$$
\begin{array}{ccc}
\left(Y \coprod_V U\right) \times \partial\Delta^n & \hookrightarrow & \left(Y \coprod_V U\right) \times \Delta^n \\
{\scriptstyle q' \times \partial\Delta^n} \big\downarrow & & \big\downarrow {\scriptstyle q' \times \Delta^n} \\
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from the pushout to the terminal corner.

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and construct the induced map

$$q^{(n)} \colon \left(Y \coprod_V U\right)^{(n)} \rightarrowtail X' \times \Delta^n$$

from the pushout to the terminal corner. It is a projective cofibration, because the projective model structure is monoidal.

Nisnevich fibrations

An objectwise fibration $E \longrightarrow F$ is a *Nisnevich fibration* if it has the right lifting property with respect to the set

$$\left\{ q^{(n)} \colon \left(Y \coprod_V U \right)^{(n)} \rightarrowtail X' \times \Delta^n \right\}_{Q, n \in \mathbb{N}}$$

where Q runs through the collection of Nisnevich squares.

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where Q runs through the collection of Nisnevich squares. (In order to ignore the empty variety $\emptyset$, add the square

$$\begin{array}{ccc} \varnothing & \rightarrowtail & \varnothing \\ \downarrow & & \downarrow \\ \varnothing & \rightarrowtail & \emptyset \end{array}$$

where $\varnothing$ is the initial space over F to this collection.)

Nisnevich fibrancy

Lemma

An objectwise fibrant space A over F is Nisnevich fibrant if and only if $A(\emptyset)$ is contractible and, for every Nisnevich square Q, the induced square

$$A(Q) = \begin{array}{ccc} A(X) & \to & A(U) \\ \downarrow & & \downarrow \\ A(Y) & \to & A(V) \end{array}$$

is a homotopy pullback square of simplicial sets.

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is a homotopy pullback square of simplicial sets.

This follows basically from the definitions and the Yoneda lemma. The lemma implies that a Nisnevich fibrant space over F and a Nisnevich square induce a Mayer-Vietoris exact sequence of homotopy groups of simplicial sets.

Nisnevich fibrancy

- Representable spaces over F are Nisnevich fibrant.
- K-theory is Nisnevich fibrant (Thomason-Trobaugh)
- A constant space L over F is Nisnevich fibrant if and only if the simplicial set L is contractible.

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- A constant space L over F is Nisnevich fibrant if and only if the simplicial set L is contractible.

Consider the representable space $\mathbb{A}^1 \setminus \{0\}$ over F. It is Nisnevich fibrant. Moreover, it is a group variety, usually denoted $\mathbb{G}_m$ or GL_1. Its values at $\mathrm{Spec}(R)$ are

$$\mathbb{G}_m(\mathrm{Spec}(R)) = R^\times$$

the units in R. Applying the classifying space functor B objectwise produces a space $\mathrm{B}\mathbb{G}_m$ over F. It is **not** Nisnevich fibrant.

Nisnevich fibrancy

In order to see this, consider the Nisnevich square

$$\begin{array}{ccc} \mathbb{A}^1 \smallsetminus \{0\} & \hookrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

which, since $\pi_1 \mathrm{B}G(X) = G(X)$ for any group space over F, induces the following sequence on homotopy groups after applying $\mathrm{B}\mathbb{G}_m$:

$$F^\times \longrightarrow F[t]^\times \times F[t^{-1}]^\times \rightarrow F[t, t^{-1}]^\times \longrightarrow \{1\}$$

It is not exact.

Nisnevich equivalences

Choose a projective cofibrant replacement functor $A^c \xrightarrow{\sim} A$ for the projective model structure. A map $f\colon A \longrightarrow B$ of spaces over F is a *Nisnevich equivalence* if, for every Nisnevich fibrant space D over F, the induced map

$$\mathbf{sSet}_{\mathbf{Spc}_F}(f^c, D)\colon \mathbf{sSet}_{\mathbf{Spc}_F}(B^c, D) \longrightarrow \mathbf{sSet}_{\mathbf{Spc}_F}(A^c, D)$$

on mapping spaces is a weak equivalence of simplicial sets.

Nisnevich equivalences

- By construction, the induced map

$$q\colon Y \coprod_V U \longrightarrow X$$

  is a Nisnevich equivalence for every Nisnevich square Q.
- The natural map $A \longrightarrow \mathrm{Nis}(A)$ to the Nisnevich sheafification is a Nisnevich equivalence for every space A over F.
- A map of smooth varieties over F is a Nisnevich equivalence if and only if it is an isomorphism of varieties.

Nisnevich homotopy theory

Theorem

The classes of Nisnevich equivalences, Nisnevich fibrations and projective cofibrations are a model structure on the category $\mathbf{Spc}_F$. *It is simplicial, combinatorial and monoidal.*

Nisnevich homotopy theory

Theorem

The classes of Nisnevich equivalences, Nisnevich fibrations and projective cofibrations are a model structure on the category $\mathbf{Spc}_F$*. It is simplicial, combinatorial and monoidal.*

The projective Nisnevich model structure is obtained by left Bousfield localization of the projective model structure on $\mathbf{Spc}_F$ with respect to the set

$$\{q\colon Y \coprod_V U \rightarrowtail X\}_Q$$

where Q runs through the collection of Nisnevich squares.

Outline

$\mathbb{A}^1$-homotopy theory

We want to use $\mathbb{A}^1$ as an interval parametrizing homotopies for varieties. However, then $\mathbb{A}^1$ should be contractible.

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We want to use $\mathbb{A}^1$ as an interval parametrizing homotopies for varieties. However, then $\mathbb{A}^1$ should be contractible.
Bousfield localize the Nisnevich projective model structure (which is too fine to study varieties anyhow) to achieve this.

$\mathbb{A}^1$-fibrancy

A Nisnevich fibrant space A over F is $\mathbb{A}^1$-*fibrant* if, for every smooth F-variety X, the map

$$A(X) \rightarrowtail A(X \times \mathbb{A}^1)$$

induced by the projection is a weak equivalence of simplicial sets.

$\mathbb{A}^1$-fibrancy

Example

The representable space $\mathbb{G}_m$ over F is $\mathbb{A}^1$-fibrant. For every $Y \in \mathrm{Sm}_F$, there is an isomorphism

$$\mathbb{G}_m(Y) = \mathrm{Hom}_{\mathrm{Sm}_F}(Y, \mathbb{G}_m) \cong \mathcal{O}_Y^\times$$

which shows that the induced map

$$\mathbb{G}_m(X) \cong \mathcal{O}_X^\times \longrightarrow \mathcal{O}_{X \times \mathbb{A}^1}^\times = \left(\mathcal{O}_X[t]\right)^\times \cong \mathbb{G}_m(X \times \mathbb{A}^1)$$

is an isomorphism of (discrete) simplicial sets.

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is an isomorphism of (discrete) simplicial sets.

Note that $(\mathbb{Z}/4\mathbb{Z})^\times \longrightarrow \left((\mathbb{Z}/4\mathbb{Z})[t]\right)^\times$ is not surjective: $(1 + 2t) \cdot (1 - 2t) = 1$

$\mathbb{A}^1$-fibrancy

Smooth curves of positive genus and abelian varieties are $\mathbb{A}^1$-fibrant as well. A “non-discrete” example is K-theory (Quillen).

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The representable space SL_2 over F is **not** $\mathbb{A}^1$-fibrant. In fact, the canonical map

$$SL_2(F) \rightarrowtail SL_2(F[t])$$

is not bijective: Not in the image is the matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ of determinant 1.

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A non-discrete example of a space over F which is not $\mathbb{A}^1$-fibrant is $\mathrm{B}\mathbb{G}_m$. The problem here, however, is the Nisnevich fibrancy, as seen before.

$\mathbb{A}^1$-equivalences

Recall that $A^c \overset{\sim}{\twoheadrightarrow} A$ is a projective cofibrant replacement functor. A map $f\colon A \longrightarrow B$ of spaces over F is an $\mathbb{A}^1$-*equivalence* if, for every $\mathbb{A}^1$-fibrant space D over F, the induced map

$$\mathbf{sSet}_{\mathbf{Spc}_F}(f^c, D)\colon \mathbf{sSet}_{\mathbf{Spc}_F}(B^c, D) \longrightarrow \mathbf{sSet}_{\mathbf{Spc}_F}(A^c, D)$$

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is a weak equivalence of simplicial sets.
A map $f\colon A \longrightarrow B$ of spaces over F is an $\mathbb{A}^1$-*fibration* if it has the right lifting property with respect to projective cofibrations which are also $\mathbb{A}^1$-equivalences.

$\mathbb{A}^1$-homotopy theory

Theorem (Morel-Voevodsky 1999)

The classes of $\mathbb{A}^1$-equivalences, $\mathbb{A}^1$-fibrations and projective cofibrations are a proper simplicial model structure on the category $\mathbf{Spc}_F$. It is combinatorial and monoidal.

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The classes of $\mathbb{A}^1$-equivalences, $\mathbb{A}^1$-fibrations and projective cofibrations are a proper simplicial model structure on the category $\mathbf{Spc}_F$*. It is combinatorial and monoidal.*

The projective $\mathbb{A}^1$-Nisnevich model structure is obtained by left Bousfield localization of the projective Nisnevich model structure on $\mathbf{Spc}_F$ with respect to the set of projections

$$\{X \times \mathbb{A}^1 \longrightarrow X\}_{X \in \mathrm{Sm}_F}.$$

The associated homotopy category $\mathbf{Ho}(F)$ is the $\mathbb{A}^1$-*homotopy category* of the field F.

$\mathbb{A}^1$-equivalences

- The canonical projection $X \times \mathbb{A}^d \longrightarrow X$ is an $\mathbb{A}^1$-equivalence by construction.
- More generally, the projection $p\colon V \longrightarrow X$ of a vector bundle is an $\mathbb{A}^1$-equivalence.

Circles in $\mathbb{A}^1$-homotopy theory

Consider the canonical covering of $\mathbb{P}^1$:

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & \hookrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

By definition of the Nisnevich homotopy theory, it is a homotopy pushout. By definition of the $\mathbb{A}^1$-homotopy theory, the corners are $\mathbb{A}^1$-contractible.

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By definition of the Nisnevich homotopy theory, it is a homotopy pushout. By definition of the $\mathbb{A}^1$-homotopy theory, the corners are $\mathbb{A}^1$-contractible. Hence $\mathbb{P}^1$ is $\mathbb{A}^1$-equivalent to the (reduced) suspension of $\mathbb{A}^1 \smallsetminus \{0\}$, pointed at 1:

$$\mathbb{P}^1 \simeq_{\mathbb{A}^1} \Sigma\big(\mathbb{A}^1 \smallsetminus \{0\}, 1\big)$$

Real and complex points

Suppose that $F \hookrightarrow \mathbb{R}$ is an embedding of fields. Taking complex resp. real analytic spaces is a left Quillen functor, hence induces functors:

$$\mathrm{real}_{\mathbb{C}} : \mathbf{Ho}(F) \longrightarrow \mathbf{Ho}(\bullet) \quad \mathrm{real}_{\mathbb{R}} : \mathbf{Ho}(F) \longrightarrow \mathbf{Ho}(\bullet)$$

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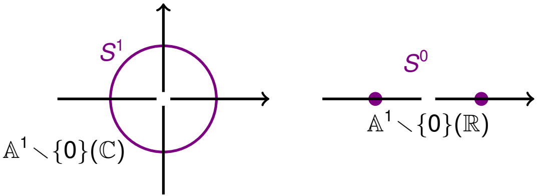
- $\mathrm{real}_{\mathbb{C}}(L) = |L| = \mathrm{real}_{\mathbb{R}}(L)$, in particular for $L = \Delta^1/\partial\Delta^1$
- $\mathrm{real}_{\mathbb{C}}(\mathbb{P}^1) = \mathbb{CP}^1 \cong S^2$, $\mathrm{real}_{\mathbb{R}}(\mathbb{P}^1) = \mathbb{RP}^1 \cong S^1$
- $\mathrm{real}_{\mathbb{C}}(\mathbb{A}^1 \setminus \{0\}) = \mathbb{C} \setminus \{0\} \simeq S^1$, $\mathrm{real}_{\mathbb{R}}(\mathbb{A}^1 \setminus \{0\}) = \mathbb{R} \setminus \{0\} \simeq S^0$



Understanding maps

One problem is to understand maps in the homotopy category. If A, B are spaces over F, the set

$$\mathrm{Hom}_{\mathbf{Ho}(F)}(A, B)$$

can be constructed as follows. Let $A^c \xrightarrow{\sim} A$ be a projective cofibrant replacement, and let $B \xrightarrow{\sim} R_{\mathbb{A}^1} B$ be an $\mathbb{A}^1$-fibrant replacement.

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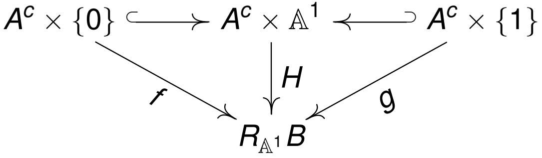
$$\mathrm{Hom}_{\mathbf{Ho}(F)}(A, B) = \mathrm{Hom}_{\mathbf{Spc}_F}(A^c, R_{\mathbb{A}^1} B)/\mathrm{homotopy}$$

is the set of equivalence classes of maps of spaces with respect to the equivalence relation given by (simplicial or $\mathbb{A}^1$) homotopy:



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The affine varieties

$$\nabla^n = \mathrm{Spec}\big(F[t_0, \ldots, t_n]/(1 - \sum_{k=0}^{n} t_k)\big)$$

assemble to a cosimplicial smooth F-variety:

$$\begin{aligned} \nabla^\bullet \colon \Delta &\longrightarrow \mathrm{Sm}_F \\ [n] &\longmapsto \nabla^n \end{aligned}$$

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Hence if $A\colon \mathrm{Sm}_F^{\mathrm{op}} \longrightarrow$ **Set** is a **Set**-valued functor,
$[n] \longmapsto A(X \times \nabla^n)$ is a simplicial set for every $X \in \mathrm{Sm}_F$.

The $\mathbb{A}^1$-singular complex

Let $A \in \mathbf{Spc}_F$. Taking its n-simplices objectwise defines

$$A_n \colon \mathrm{Sm}_F^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

for every $n \in \mathbb{N}$. The space $\mathrm{Sing}_{\mathbb{A}^1}(A)$ over F has as its n-simplices the following functor:

$$\begin{aligned}\big(\mathrm{Sing}_{\mathbb{A}^1}(A)\big)_n \colon \mathrm{Sm}_F^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ X &\longmapsto A_n(X \times \nabla^n)\end{aligned}$$

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This is in fact the diagonal of a bisimplicial presheaf on Sm_F. Suggestive notation:

$$\mathrm{Sing}_{\mathbb{A}^1}(A) = A(- \times \nabla^\bullet)$$

The $\mathbb{A}^1$-singular complex

Lemma

Let $A \in \mathbf{Spc}_F$. The canonical morphisms $X \cong X \times \nabla^0 \longrightarrow X \times \nabla^n$ induce a natural map

$$A \longrightarrow \mathrm{Sing}_{\mathbb{A}^1}(A)$$

which is an $\mathbb{A}^1$-equivalence. Moreover, for every $X \in \mathrm{Sm}_F$, the map

$$\mathrm{Sing}_{\mathbb{A}^1}(A)(X) \longrightarrow \mathrm{Sing}_{\mathbb{A}^1}(A)(X \times \mathbb{A}^1)$$

is a weak equivalence of simplicial sets.

The $\mathbb{A}^1$-singular complex

Proof.

The map $A \longrightarrow \mathrm{Sing}_{\mathbb{A}^1}(A)$ is an $\mathbb{A}^1$-homotopy equivalence. An $\mathbb{A}^1$-homotopy $\mathrm{Sing}_{\mathbb{A}^1}(A) \times \mathbb{A}^1 \longrightarrow \mathrm{Sing}_{\mathbb{A}^1}(A)$ corresponds to a map

$$A(- \times \nabla^\bullet) \longrightarrow A(- \times \nabla^\bullet \times \mathbb{A}^1).$$

Multiplication with t (where $\mathbb{A}^1 = \mathrm{Spec}\big(F[t]\big)$) is an $\mathbb{A}^1$-homotopy from $\mathrm{id}_{\nabla^\bullet}$ to the constant map with value ∇^0. This implies the first statement. The second statement proceeds similarly. □

Path components

If $A \in \mathbf{Spc}_F$, let $\pi_0 A$ be the composition $\pi_0 \circ A$, where

$$\pi_0 : \mathbf{sSet} \longrightarrow \mathbf{Set}$$

sends a simplicial set to the set of its path components.

Theorem

Let $A \in \mathbf{Spc}_F$ be Nisnevich fibrant. If $\pi_0 A(X) \longrightarrow \pi_0 A(X \times \mathbb{A}^1)$ is bijective for every $X \in \mathrm{Sm}_F$, then $\mathrm{Sing}_{\mathbb{A}^1}(A)$ is $\mathbb{A}^1$-fibrant.

Path components and $\mathbb{A}^1$-fibrancy

Theorem ("Singular")

Let $A \in \mathbf{Spc}_F$ be Nisnevich fibrant. If $\pi_0 A(Z) \longrightarrow \pi_0 A(Z \times \mathbb{A}^1)$ is bijective for every $Z \in \mathrm{Sm}_F$, then $\mathrm{Sing}_{\mathbb{A}^1}(A)$ is $\mathbb{A}^1$-fibrant.

Proof.

By the Lemma above, it remains to prove that $\mathrm{Sing}_{\mathbb{A}^1}(A)$ is Nisnevich fibrant. Product with a smooth F-variety preserves Nisnevich squares. Hence $\mathrm{Sing}_{\mathbb{A}^1}(A)$, applied to a Nisnevich square Q, is the diagonal of a square of bisimplicial sets

$$\begin{array}{ccc} A(X \times \nabla^\bullet) & \rightarrowtail & A(U \times \nabla^\bullet) \\ \downarrow & & \downarrow \\ A(Y \times \nabla^\bullet) & \rightarrowtail & A(V \times \nabla^\bullet) \end{array}$$

which is a homotopy pullback square whenever $\bullet$ is replaced by some $n \in \mathbb{N}$. $\square$

Path components and $\mathbb{A}^1$-fibrancy

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which is a homotopy pullback square whenever $\bullet$ is replaced by some $n \in \mathbb{N}$. For every Z occurring in Q, $[n] \longmapsto \pi_0 A(Z \times \nabla^n)$ is a discrete simplicial set. A result of Bousfield and Friedlander from 1978 implies that the diagonal square is a homotopy pullback square. □

Path components and $\mathbb{A}^1$-fibrancy

To obtain an $\mathbb{A}^1$-fibrant replacement for an arbitrary space over F, let $A \xrightarrow{\sim} R_{\mathrm{Nis}}A$ be a Nisnevich fibrant replacement. Then the colimit of the sequence

$$A \longrightarrow R_{\mathrm{Nis}}A \longrightarrow \mathrm{Sing}_{\mathbb{A}^1}(R_{\mathrm{Nis}}A) \longrightarrow R_{\mathrm{Nis}}\,\mathrm{Sing}_{\mathbb{A}^1}(R_{\mathrm{Nis}}A) \longrightarrow \cdots$$

is an $\mathbb{A}^1$-fibrant replacement of A.

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is an $\mathbb{A}^1$-fibrant replacement of A.
One consequence of this fibrant replacement is that the canonical map

$$A_0\big(\mathrm{Spec}(F)\big) \longrightarrow \mathrm{Hom}_{\mathbf{Ho}(F)}\big(\mathrm{Spec}(F), A\big)$$

is surjective for every space A over F. In particular, having a rational point is an invariant of the $\mathbb{A}^1$-homotopy category.

Outline

$\mathbb{A}^1$-homotopy types of classifying spaces

Let $G\colon \mathrm{Sm}_F^{\mathrm{op}} \longrightarrow$ **Grp** be a presheaf of groups (such as GL_n). Let $\mathrm{B}G\colon \mathrm{Sm}_F^{\mathrm{op}} \longrightarrow$ **Spc** denote the induced *classifying space* of G. Inclusion of 1-simplices defines a natural pointed map

$$\Sigma G \longrightarrow \mathrm{B}G$$

where Σ is the reduced suspension (note that both G and $\mathrm{B}G$ have canonical base points).

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Lemma

The adjoint map

$$\mathrm{GL}_n \longrightarrow \Omega R_{\mathrm{Nis}} \mathrm{B}\,\mathrm{GL}_n$$

is a Nisnevich equivalence. Moreover, $\pi_0 R_{\mathrm{Nis}} \mathrm{B}\,\mathrm{GL}_n(X)$ is naturally isomorphic to the set of isomorphism classes of vector bundles of rank n over X.

$\mathbb{A}^1$-homotopy types of classifying spaces

By construction, $R_{\mathrm{Nis}}\mathrm{B}\,\mathrm{GL}_n$ is Nisnevich fibrant. Is it $\mathbb{A}^1$-fibrant?

$\mathbb{A}^1$-homotopy types of classifying spaces

By construction, $R_{\mathrm{Nis}}\mathrm{B}\,\mathrm{GL}_n$ is Nisnevich fibrant. Is it $\mathbb{A}^1$-fibrant? No. Consider $\mathbb{P}^1$. There is a vector bundle on $\mathbb{P}^1 \times \mathbb{A}^1$ which is not isomorphic to the pullback along $\mathbb{P}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{P}^1$ of a vector bundle on $\mathbb{P}^1$.

$\mathbb{A}^1$-homotopy types of classifying spaces

In order to describe such a vector bundle, use the standard open covering of $\mathbb{P}^1$ by $\mathbb{A}^1 = \mathrm{Spec}(F[t])$ and $\mathbb{A}^1 = \mathrm{Spec}(F[t^{-1}])$. It induces an open cover of $\mathbb{P}^1 \times \mathrm{Spec}(F[x])$. A rank 2 vector bundle on $\mathbb{P}^1 \times \mathbb{A}^1$ is then specified by a linear glueing isomorphism

$$\mathbb{A}^2 \times \mathrm{Spec}\big(F[t, t^{-1}, x]\big) \longrightarrow \mathbb{A}^2 \times \mathrm{Spec}\big(F[t, t^{-1}, x]\big)$$

over $\mathrm{Spec}\big(F[t, t^{-1}, x]\big)$.

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over $\mathrm{Spec}\big(F[t, t^{-1}, x]\big)$. Using the invertible matrix

$$\begin{pmatrix} t^{-1} & xt \\ 0 & 1 \end{pmatrix}$$

produces a vector bundle $V \longrightarrow \mathbb{P}^1 \times \mathbb{A}^1$ such that

$$V|_{\mathbb{P}^1 \times \{0\}} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$$
$$V|_{\mathbb{P}^1 \times \{1\}} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

These vector bundles are not isomorphic by Grothendieck's theorem.

$\mathbb{A}^1$-invariance for vector bundles

Theorem (Lindel 1982)

Let $X = \mathrm{Spec}(R)$ be a smooth affine F-variety. Then pullback along the projection $X \times \mathbb{A}^1 \longrightarrow X$ induces a bijection on the set of isomorphism classes of vector bundles of rank n.

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Hence $R_{\mathrm{Nis}} \mathrm{B}\,\mathrm{GL}_n$ satisfies the condition of the "singular" Theorem when restricted to smooth affine F-varieties.

From smooth to smooth affine varieties and back

Theorem ("Affine")

Let A be a space over F. Suppose that A sends every Nisnevich square of smooth affine F-varieties to a homotopy pullback square. Suppose further that

$$A(X \times \mathbb{A}^1 \xrightarrow{\text{pr}} X)$$

is a weak equivalence for every smooth affine F-variety X. Then $R_{\text{Nis}} \operatorname{Sing}_{\mathbb{A}^1}(A)$ is $\mathbb{A}^1$-fibrant, and the canonical map

$$A(X) \longrightarrow R_{\text{Nis}} \operatorname{Sing}_{\mathbb{A}^1}(A)(X)$$

is a weak equivalence of simplicial sets for every smooth affine F-variety X.

A first representability result

Corollary

Let $X = \mathrm{Spec}(R)$ be a smooth affine F-variety. Then there is a bijection

$$\mathrm{Hom}_{\mathbf{Ho}(F)}(X, \mathrm{B}GL_n) \cong \{\text{rank } n \text{ vector bundles over } X\}/\text{iso}$$

which is natural in X.

The proof uses the “affine” Theorem, which relies on Lindel’s Theorem. The proof of the “affine” Theorem relies on the “singular” Theorem.

More on classifying spaces

One construction of $\mathrm{B}G$ proceeds as follows. Let $\mathrm{E}G$ be the space whose n-simplices are given by the $n+1$-fold product

$$(\mathrm{E}G)_n = G \times G \times \cdots \times G$$

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The “extra” factor of G allows G to act freely, such that the quotient $\mathrm{E}G/G \cong \mathrm{B}G$. Moreover $\mathrm{E}G$ is contractible as the classifying space of a category with an initial object. More generally, any G-space over F whose underlying space is contractible has orbit space weakly equivalent to $\mathrm{B}G$.

A geometric representability result

Consider the variety $\mathrm{Lin}(n, k)$ of linear embeddings $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+k}$. The group GL_{n+k} acts on the target of these embeddings, hence on $\mathrm{Lin}(n, k)$. Via this action, $\mathrm{Lin}(n, k)$ is a homogeneous space for GL_{n+k}.

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The group GL_n over F acts on the source, hence on $\mathrm{Lin}(n,k)$. This action is free, and the quotient variety is $\mathrm{Gr}(n,k)$, the variety of linear subspaces.

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The group GL_n over F acts on the source, hence on $\mathrm{Lin}(n, k)$. This action is free, and the quotient variety is $\mathrm{Gr}(n, k)$, the variety of linear subspaces.
The morphism $x \longmapsto (0, x)$ induces a closed embedding

$$\mathrm{Lin}(n, k) \hookrightarrow \mathrm{Lin}(n, n + k)$$

For every $k \in \mathbb{N}$ there is an $\mathbb{A}^1$-homotopy

$$\mathrm{Lin}(n, k) \times \mathbb{A}^1 \longrightarrow \mathrm{Lin}(n, n + k)$$

from this closed embedding to the constant morphism with value the standard linear embedding.

A geometric representability result

In particular, the colimit $\mathrm{Lin}(n, \infty)$ is $\mathbb{A}^1$-contractible and carries a free GL_n-action whose quotient homogeneous space is $\mathrm{Gr}(n, \infty)$. Hence

$$\mathrm{B\,GL}_n \simeq_{\mathbb{A}^1} \mathrm{Gr}(n, \infty)$$

and the Theorems above imply the following.

Theorem (Morel 2012)

Let F be a field, and let $X = \mathrm{Spec}(R)$ be a smooth affine F-variety. Sending a map $f\colon X \longrightarrow \mathrm{Gr}(n, \infty)$ to the pullback vector bundle $f^*\gamma_{n,\infty}$ defines a bijection

$$\mathrm{Hom}_{\mathbf{Ho}(F)}(X, \mathrm{Gr}(n, \infty)) \cong \{\text{rank } n \text{ vector bundles over } X\}/\text{iso}$$

from the set of $\mathbb{A}^1$-homotopy classes of maps to the set of isomorphism classes of vector bundles of rank n.

Outline

Applications

One may apply Morel's classifying theorem as follows. The inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ adding 1 as last diagonal entry induces a map

$$\mathrm{Gr}(n,\infty) \longrightarrow \mathrm{Gr}(n+1,\infty)$$

of spaces over F whose $\mathbb{A}^1$-homotopy fiber is the homogeneous variety $\mathrm{GL}_{n+1} / \mathrm{GL}_n$. The latter can be identified up to $\mathbb{A}^1$-equivalence with $\mathbb{A}^{n+1} \smallsetminus \{0\}$, supplying an $\mathbb{A}^1$-homotopy fiber sequence:

$$\mathbb{A}^{n+1} \smallsetminus \{0\} \longrightarrow \mathrm{Gr}(n,\infty) \longrightarrow \mathrm{Gr}(n+1,\infty)$$

$\mathbb{A}^1$-connectivity

Theorem (Morel 2012)

For every $n \in \mathbb{N}$, the space $\mathbb{A}^{n+1} \setminus \{0\}$ over F is $n-1$-connected. Moreover, the first nontrivial homotopy group

$$\pi_n R_{\mathbb{A}^1}\left(\mathbb{A}^{n+1} \setminus \{0\}\right)(\mathrm{Spec}(F))$$

is the n-th Milnor-Witt K-group of F.

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The preceding $\mathbb{A}^1$-homotopy fiber sequence and this connectivity result show that every vector bundle over a smooth affine F-variety X whose rank exceeds the dimension of X splits off a trivial line bundle. A more refined version allows an obstruction theory via Euler classes – see Morel (2012), and recent results of Asok and Fasel solving cases of Murthy's conjecture.

Murthy's conjecture

Conjecture (Murthy 1999)

Let F be an algebraically closed field, let $X = \mathrm{Spec}(R)$ be a smooth affine F-variety of dimension d, and let $V \longrightarrow X$ be a vector bundle of rank $d-1$. Then $V \longrightarrow X$ splits off a trivial line bundle if and only if the Chern class $c_{d-1}(V \longrightarrow X) \in CH^{d-1}(X)$ is zero.

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The conjecture is tautological for surfaces ($d = 2$).

Theorem (Asok-Fasel 2014)

Let F be an algebraically closed field of characteristic not 2. Then Murthy's conjecture holds for $d \in \{3, 4\}$.

Milnor-Witt K-theory (Hopkins-Morel)

For any field F, let $K_*^{MW}(F)$ denote the graded associative ring generated by elements $[a], a \in F \smallsetminus \{0\}$, of degree 1 and an element η of degree -1, subject to the following relations:

1. $[a] \cdot [1 - a] = 0$ for all $a \in F \smallsetminus \{0, 1\}$
2. $[ab] = [a] + [b] + \eta \cdot [a] \cdot [b]$ for all $a, b \in F \smallsetminus \{0\}$:
3. $[a] \cdot \eta = \eta \cdot [a]$ for all $a \in F \smallsetminus \{0\}$:
4. $\eta \cdot (\eta \cdot [-1] + 1) = -\eta$

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In particular, $K_0^{MW}(F)$ is the Grothendieck-Witt ring of symmetric bilinear forms, and $K_{-n}^{MW}(F)$ is isomorphic to the Witt ring of symmetric bilinear forms for $n > 0$.

Outline

Stable $\mathbb{A}^1$-homotopy theory

Morel's connectivity theorem can be interpreted as the computation of the zeroth $\mathbb{P}^1$-stable homotopy groups of spheres.

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Why $\mathbb{P}^1$-stable? $\mathbf{Ho}(F)$ is complicated. Stabilizing simplifies the structure.
Stabilizing *with respect to* $\mathbb{P}^1$ is recommended for geometric reasons (Homotopy Purity Theorem).

$\mathbb{P}^1$-spectra

A $\mathbb{P}^1$-*spectrum* over F is:

- $E = (E_0, E_1, \dots, E_n \dots)$ and structure maps $\mathbb{P}^1 \wedge E_n \longrightarrow E_{n+1}$, where
- $E_n \colon \mathrm{Sm}_F^{\mathrm{op}} \longrightarrow \mathbf{sSet}_\bullet$ is a pointed space over F for all $n \in \mathbb{N}$.

The smash product of pointed spaces over F is $B \wedge C = B \times C / B \vee C$.

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The smash product of pointed spaces over F is $B \wedge C = B \times C / B \vee C$.
Maps of $\mathbb{P}^1$-spectra are the obvious ones, giving a category $\mathbf{Spt}_{\mathbb{P}^1}(F)$.

$\mathbb{P}^1$-suspension spectra

Example

Any smooth F-scheme X (which may not have a rational point) defines a $\mathbb{P}^1$-suspension spectrum

$$\Sigma^\infty X_+ = (X_+, \mathbb{P}^1 \wedge X_+, \mathbb{P}^1 \wedge \mathbb{P}^1 \wedge X_+, \dots)$$

with identities as structure maps, where $X_+ = X \coprod \mathrm{Spec}(F)$.

An algebraic K-theory $\mathbb{P}^1$-spectrum

Voevodsky's algebraic K-theory $\mathbb{P}^1$-spectrum:

- $\mathbf{KGL} = (K^Q, K^Q \dots)$ with structure map
- $\mathbb{P}^1 \wedge \mathbf{KGL}_n = \mathbb{P}^1 \wedge K^Q \xrightarrow{\beta} K^Q = \mathbf{KGL}_{n+1}$,
- where β is multiplication with the Bott element $[\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\mathbb{P}^1}(-1)] \in K^0(\mathbb{P}^1)$.

Outline

Model structures on $\mathbb{P}^1$-spectra

Let $f\colon D \longrightarrow E$ be a map of $\mathbb{P}^1$-spectra.

- It is a *strict equivalence* if f_n is an $\mathbb{A}^1$-equivalence of spaces over F for every $n \in \mathbb{N}$.
- It is a *strict fibration* if f_n is an $\mathbb{A}^1$-fibration of spaces over F for every $n \in \mathbb{N}$.
- It is a *cofibration* if f_n and the induced map $D_{n+1} \coprod_{\mathbb{P}^1 \wedge D_n} \mathbb{P}^1 \wedge E_n \longrightarrow E_{n+1}$ is a projective cofibration for every $n \in \mathbb{N}$.

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Theorem

These classes are a model structure on $\mathbf{Spt}_{\mathbb{P}^1}(F)$*. It is simplicial and combinatorial.*

Model structures on $\mathbb{P}^1$-spectra

This model structure is not the right one. Use the functor

$$\mathrm{Q}(E) = \operatorname{colim}\big(E \longrightarrow \Omega_{\mathbb{P}^1} R_{\mathbb{A}^1} E_{+1} \longrightarrow \Omega^2_{\mathbb{P}^1} R_{\mathbb{A}^1} E_{+2} \longrightarrow \cdots\big)$$

as in topology.

Model structures on $\mathbb{P}^1$-spectra

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as in topology.
A strict fibration $f\colon D \rightarrowtail E$ is a $\mathbb{P}^1$-*stable fibration* if

$$\begin{array}{ccc} D & \rightarrowtail & \mathrm{Q}(D) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Q}(f)} \\ E & \rightarrowtail & \mathrm{Q}(E) \end{array}$$

is a homotopy pullback square. A map f is a $\mathbb{P}^1$-*stable equivalence* if $\mathrm{Q}(f)$ is a strict equivalence.

The $\mathbb{P}^1$-stable homotopy category

Theorem (Voevodsky 1998)

$\mathbb{P}^1$-stable equivalences, $\mathbb{P}^1$-stable fibrations and cofibrations form a model structure on $\mathbf{Spt}_{\mathbb{P}^1}(F)$. It is simplicial and combinatorial.

Its homotopy category is the $\mathbb{P}^1$-stable homotopy category $\mathbf{SH}(F)$. It is triangulated, and $\mathbb{P}^1 \wedge -$ is an equivalence on it.

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Theorem (Levine 2012)

The classical stable homotopy category fully embeds in $\mathbf{SH}(\mathbb{C})$

The zeroth line

Set $\mathbf{1} = \mathrm{Spec}(F)_+$, $S^{2,1} = \mathbb{P}^1$, $S^{1,1} = \mathbb{A}^1 \smallsetminus \{0\}$ and $S^{1,0} = S^1$ as objects in $\mathbf{SH}(F)$. Then since $S^{2,1} \wedge - \colon \mathbf{SH}(F) \longrightarrow \mathbf{SH}(F)$ is an equivalence, $S^{p,q}$ is defined for all $p, q \in \mathbb{Z}$. If $E \in \mathbf{SH}(F)$, set

$$\pi_{p,q}E = \mathrm{Hom}_{\mathbf{SH}(F)}(S^{p,q}, E)$$

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Theorem (Morel)

Let F be a field and $n \in \mathbb{Z}$. There is an isomorphism

$$\pi_{n,n}\mathbf{1} \cong K^{MW}_{-n}(F)$$

of graded rings.

The first line

The next nontrivial $\mathbb{P}^1$-stable homotopy group of spheres is known over fields of characteristic zero, up to a completion at the element η.

Theorem (R.-Spitzweck-Østvær)

Let F be a field of characteristic zero. For every integer n, the unit map $\mathbf{1} \longrightarrow \mathbf{KQ}$ defines a short exact sequence:

$$0 \longrightarrow K^{\text{Milnor}}_{2-n}/24 \longrightarrow \pi_{n+1,n}\mathbf{1}^{\wedge}_{\eta} \longrightarrow \pi_{n+1,n}\mathbf{KQ}$$

If $n \geq -3$, the map on the right hand side is surjective. In particular, $\pi_{3,2}\mathbf{1}^{\wedge}_{\eta} \cong \mathbb{Z}/24$ and $\pi_{n+1,n}\mathbf{1}^{\wedge}_{\eta} \cong 0$ for $n \geq 3$.

Here $\pi_{n+1,n}\mathbf{KQ}$ is a hermitian K-theory group of F.

Exercises

- Show that the tautological vector bundle over a projective space (or a Grassmannian) is a vector bundle.
- Let L/F be a field extension, and let $\{U \hookrightarrow X, Y \longrightarrow X\}$ be a Nisnevich covering obtained from a Nisnevich square. Show that every morphism $\mathrm{Spec}(L) \longrightarrow X$ lifts either to U or to Y.
- The proof of the Homotopy Purity Theorem proceeds via a blow-up construction, which implies $\mathbb{A}^1$-equivalences:

$$\frac{X}{X \smallsetminus i(Z)} \xrightarrow{\sim} \frac{\mathrm{Bl}(X \times \mathbb{A}^1, Z \times \{0\})}{\mathrm{Bl}(X \times \mathbb{A}^1, Z \times \{0\}) \smallsetminus i'(Z \times \mathbb{A}^1)} \xleftarrow{\sim} \frac{Ni}{Ni - z(Z)}$$

  Provide these $\mathbb{A}^1$-equivalences in the special case $Z = \{0\} \hookrightarrow \mathbb{A}^d = X$, using that the blow-up $\mathrm{Bl}(\mathbb{A}^d \times \mathbb{A}^1, \{0\} \times \{0\})$ of $\{0\} \hookrightarrow \mathbb{A}^{d+1}$ is the total space of the tautological line bundle over $\mathbb{P}^d$.
- Prove the Lemma characterizing $\mathbb{A}^1$-fibrant objects.

Exercises

- Verify that $\nabla^\bullet \colon \Delta \longrightarrow \mathrm{Sm}_{\mathbb{Z}}$ is a cosimplicial smooth $\mathbb{Z}$-variety.
- Prove the Lemma on the $\mathbb{A}^1$-singular complex.
- Prove that the colimit of a filtered diagram of $\mathbb{A}^1$-fibrant spaces over F is again $\mathbb{A}^1$-fibrant.
- Let n be a positive natural number. Show that $\mathbb{A}^2 \setminus \{0\}$ is $\mathbb{A}^1$-equivalent to the reduced suspension $\Sigma\big(\mathbb{A}^1 \setminus \{0\} \wedge \mathbb{A}^1 \setminus \{0\}\big)$. What about $\mathbb{A}^n \setminus \{0\}$?
- Show that forgetting the last column is a morphism

$$\mathrm{SL}_2 \longrightarrow \mathbb{A}^2 \setminus \{0\}$$

which is an $\mathbb{A}^1$-equivalence. What about $\mathrm{SL}_n \longrightarrow \mathbb{A}^n \setminus \{0\}$?

Exercises

- Show that $\mathbb{A}^\infty \smallsetminus \{0\}$ is $\mathbb{A}^1$-contractible.
- Is the complement of the zero section of the tautological line bundle on $\mathbb{P}^\infty$ also $\mathbb{A}^1$-contractible?
- Let F be a field. The *Milnor K-theory* $K^{\mathrm{Mil}}(F)$ of F is the quotient of the tensor algebra $\oplus_{n\in\mathbb{N}}(F \smallsetminus \{0\})^{\otimes n}$ by the ideal generated by the set $\{a \otimes (1-a)\}_{a\in F\smallsetminus\{0,1\}}$. Compute the Milnor K-theory of finite fields. Show that $K_n^{\mathrm{Mil}}(\mathbb{R})$ is the direct sum of a cyclic group of order two and a divisible group if $n > 0$.